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Invariants of solvable Lie algebras of dimension six

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Abstract. All invariant functions of the coadjoint representation are determined for real solvable Lie algebras of dimension six having a nilradical of dimension four. Many questions related to the nature of these invariants are analysed. In particular, we derive the general type of functions in terms of which all the invariants can be expressed, and give the characterization of all solvable Lie algebras with Abelian nilradical that have no non-trivial invariants. The existence of Casimir operators among these invariants is also investigated.

1. Introduction

Invariant functions of semisimple Lie algebras were determined long ago. Racah [1] pursuing a work undertaken by H B G Casimir, B L V der Warden and some other physicists, was able to give in the 1950s an explicit construction for the invariants of semisimple Lie algebras. They are all polynomial and hence Casimir operators and their number equals the dimension of the Cartan subalgebra. Moreover, they can all be chosen to be homogeneous symmetric polynomials in their generators.

For solvable Lie algebras, neither the number of invariants, nor the specific type of functions in terms of which they can be expressed is known. Some effort have been made in this direction in recent literature, but they only led to a partial characterization of the invariants for some families of solvable Lie algebras, including all solvable Lie algebras of low dimension not exceeding five [2–4].

Invariants have been determined only for a small number of non-semisimple groups. They are well known for groups such as the Poincaré group [5], the Euclidean group E(3) and the Galilei group [6]. This determination has also been done for all subgroups of the Poincaré and similitude groups of the four- and three-dimensional Minkowski space and for all subgroups of the $O(4, 1)$ de Sitter group, and the physical meaning of these invariants have been discussed [7, 8].

The determination of the invariant functions of Lie algebras (or Lie groups equivalently) is motivated by the important role played by these functions in physics, in representation theory or in group analysis of differential equations [9–11]. In particular, Casimir invariants of Lie algebras can be used to label irreducible representations [9]. In physics, invariant operators of dynamical groups characterize specific properties of physical systems by providing mass formulae and energy spectra [7, 8].

As regards the determination of the invariants of an arbitrary Lie algebra, and in the context of all the results presently available on this question, the determination of the invariants of solvable Lie algebras is the most needed. Indeed, the invariants of semisimple Lie algebras are all known and on the strength of the Levi decomposition theorem, any finite-dimensional Lie

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algebra can be written as a semidirect sum of a solvable ideal and a semisimple subalgebra. The great utility of non-polynomial invariants clearly appears, for example, in the study of integrable Hamiltonian systems. Fomenko and Trofimov show in [10] that integrating Hamiltonian systems on Lie algebras, most of which are required to be solvable, can be reduced to solving a system of linear first-order partial differential equations (PDEs) of the form $\sum_{k,j} C_{ij}^k x_j(\partial F/\partial x_j) = 0$, i.e. to the determination of the invariants of the coadjoint representation. Further discussion on the applications of the invariants of Lie groups can be found in recent scientific literature close to mathematical physics [12–14].

In this paper we determine explicitly all invariant functions of the coadjoint representation of all equivalence classes of indecomposable solvable and non-nilpotent Lie algebras N_6 of dimension six over $\mathbb{R}$, having a nilradical of dimension four. The list of all of these Lie algebras was given recently by Turkowski [15] and consists of a total of 40 classes of Lie algebras. This list completes the classification of all non-nilpotent solvable Lie algebras of dimension six started by Mubarakzyanov [16]. Nilpotent Lie algebras of dimension six were classified by Morozov [17]. Invariants of all real Lie algebras of dimension not exceeding five and of nilpotent Lie algebras of dimension six were determined in [2]. Since the nilradical $\mathcal{M}$ of any solvable Lie algebra L must satisfy $\dim \mathcal{M} \geqslant \frac{1}{2}\dim L$, for six-dimensional solvable Lie algebras there are only four cases to consider: nilpotent six-dimensional algebras and solvable Lie algebras that contain five-, four- and three-dimensional nilradicals. Algebras N_6 that possess five-dimensional nilradicals were classified by Mubarakzyanov [16] into 99 classes. However, the determination of the invariants of these Lie algebras for which $\dim L/\mathcal{M} = 1$ is simpler and particularly straightforward when the nilradical is Abelian. Furthermore, they are likely to be similar to the invariants of five-dimensional solvable Lie algebras with nilradicals of dimension four computed in [2]. Algebras N_6 that contain three-dimensional nilradicals are decomposable [16]. Consequently, we restrict our determination to solvable Lie algebras N_6 having a nilradical of dimension four.

2. Basic definitions and results

Let G be a connected Lie group and denote by $L = L(G)$ its Lie algebra, and by L^* the dual space of L.

Definition 1.

(a) The map

$$\mathrm{Ad}^*: G \to GL(L^*): (\mathrm{Ad}_g^* f)(x) = f(\mathrm{Ad}_{g^{-1}} x)$$

for all $g \in G$, $f \in L^*$ and $x \in L$ is called the coadjoint representation of the Lie group G.

(b) A function $F \in C^\infty(L^*)$ is called an invariant of the coadjoint representation if $F(\mathrm{Ad}_g^* f) = F(f)$, for all $g \in G$ and $f \in L^*$.

(c) A fundamental set of invariants of L is a subset of $C^\infty(L^*)$ consisting of a maximal number of functionally independent invariants of L.

Let $\mathcal{B} = \{V_1, \ldots, V_n\}$ be a basis of the finite- and n-dimensional solvable Lie algebra L, and let $(v_1, \ldots, v_n)$ be a coordinate system in L^* associated with the dual basis of $\mathcal{B}$. Let $\tilde{V}_i = (\mathrm{Ad}^*)_*(V_i)$ be the infinitesimal generator of the group action Ad^* corresponding to V_i, where $(\mathrm{Ad}^*)_*$ is the corresponding action of L on L^*. We have,

$$\tilde{V}_i(f) = (\mathrm{Ad}^*)_*(V_i)(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}^*_{\exp tVi}\, f$$

for all $f \in L^*$. The following result give some of the properties of the infinitesimal generators of the coadjoint representation.

Theorem 1.

(a) A function $F \in C^\infty(L^*)$ is an invariant of the coadjoint representation if and only if

$$\tilde{V}_i \cdot F = 0 \qquad \textit{for all} \quad i = 1, \ldots, n \tag{2.1}$$

(b)
$$\tilde{V}_i = -\sum_j \sum_k c_{ij}^k v_k \frac{\partial}{\partial v_j} \qquad \textit{and} \qquad [\tilde{V}_i, \tilde{V}_j] = \sum_k c_{ij}^k \tilde{V}_k \tag{2.2}$$

where $[V_i, V_j] = \sum_k c_{ij}^k V_k$, i.e. the c_{ij}^k's $(i, j, k = 1, \ldots, n)$ are the structure constants of L.

The invariant functions of the coadjoint representation of a Lie algebra are commonly just called invariants and we shall use these two terminologies interchangeably. It is also clear from (2.1) and (2.2) that the invariants are given as solutions of a system of linear first-order partial differential equations. This is indeed the method that we shall use for their determination and it has been used by many other authors [2, 6, 18, 19].

For any matrix $A = (a_k^j)$ of an endomorphism of a vector space V over $\mathbb{R}$, define the vector field $\partial(A)$ on V^*, the dual space of V, by

$$\partial(A) = -\sum_{j=1}^{n} \left(\sum_{j=1}^{n} a_k^j v_k \right) \partial_{v_j}$$

where $(v_1, \ldots, v_n)$ is a coordinate system on V^*. Consequently, if A^u represents the matrix of the adjoint operator Ad V_u $(u = 1, \ldots, n)$ of L, according to the form of the infinitesimal generators $\tilde{V}_u$ given by equation (2.2), we have $\partial(A^u) = \tilde{V}_u$ and the invariants of L are the solutions to the system of differential equations

$$\partial(A^u) \cdot F = 0 \qquad (i = 1, \ldots, n).$$

The invariants of L are thus completely determined by the matrices A^u of the adjoint operators of L.

Denote by $S(L^*)$ and $S = S(L)$ the symmetric algebras of L^* and L, respectively. They are all isomorphic to the ring $\mathbb{R}[X_1, \ldots, X_n]$ of polynomials in n indeterminates over $\mathbb{R}$, and hence $S(L^*) \simeq S$. Consequently, a polynomial invariant can be viewed as a function on L, and by extension invariants of the coadjoint representation are all seen as functions on L and called generalized Casimir invariants. This means that in expression (2.2) of the infinitesimal generators, we shall consider $(v_1, \ldots, v_n)$ as a coordinate system on L rather than on L^*. Let $\mathfrak{A} = \mathfrak{A}(L)$ denote the universal enveloping algebra of L. Set

$$\mathfrak{A}^I = \{u \in \mathfrak{A}: [V_i, u] = 0, \forall i\}$$
$$S^I = \{p \in S: \tilde{V}_i(p) = 0, \forall i\}.$$

Then $\mathfrak{A}^I$ is the centre of $\mathfrak{A}$, i.e. the set of all Casimir operators of L, while S^I is simply the set of all polynomial invariants of the coadjoint representation. Since the algebras $\mathfrak{A}$ and S are both the Nœtherian ring without divisors of zero, we can construct their quotient field $D(\mathfrak{A})$ and $D(S)$, respectively. The adjoint representation is then extended to $D(\mathfrak{A})$ by setting $[r_1, r_2] = r_1 r_2 - r_2 r_1$ for all $r_1, r_2 \in D(\mathfrak{A})$. This allows us to define two new sets $D(\mathfrak{A})^I$ and $D(S)^I$ by

$$D(\mathfrak{A})^I = \{u \in D(\mathfrak{A}): [V_i, u] = 0, \forall i\}$$
$$D(S)^I = \{p \in D(S): \tilde{V}_i \cdot p = 0, \forall i\}.$$

Definition 2. Elements of $D(\mathfrak{A})^I$ are called rational Casimir invariants.

It is clear that elements of $D(S)^I$ are rational invariant functions of the coadjoint representation. We also note that $D(\mathfrak{A})^I$ contains $D(\mathfrak{A}^I)$, the quotient field of $\mathfrak{A}^I$, and thus it contains all polynomial invariants and their quotients.

Theorem 2.

(a) *The Abelian algebras* $\mathfrak{A}^I$ *and* S^I *are algebraically isomorphic.*
(b) *The fields* $D(S)^I$ *and* $D(\mathfrak{A})^I$ *are algebraically isomorphic.*

Part (a) of this theorem is a result of [20, 21], while part (b) is proven in [22]. It is part (a) of this theorem that allows us to identify each polynomial invariant of the coadjoint representation with a Casimir operator. Moreover, it implies that the transcendence degrees of S^I and $\mathfrak{A}^I$ over $\mathbb{R}$ are identical, which means that the cardinal τ of a maximal set of algebraically independent elements in S^I is the same as for $\mathfrak{A}^I$. Similarly, the cardinal τ' of a maximal set of algebraically independent elements in $D(S)^I$ is the same as for $D(\mathfrak{A})^I$.

We return to the Lie algebra L itself and denote by $M_L = (M_L^{ij})$ the matrix representing the commutator table of L, i.e. having as entries the polynomial functions $M_L^{ij} = \sum_k c_{ij}^k v_k$. Let $R(L) = \operatorname{rank}(M_L) = \operatorname{Sup}_{v_1,\ldots,v_n} \operatorname{rank}(c_{ij}^k v_k)$.

Theorem 3. *The maximal number* $\mathcal{N}$ *of functionally independent invariants of the coadjoint representation is given by the equality* $\mathcal{N} = \dim L - R(L)$.

Proof. It is indeed well known [23] that the system of PDEs

$$\left(\sum_j f_{ij} \frac{\partial}{\partial y_j}\right)_. F = 0 \qquad (i = 1, \ldots, m)$$

where $(y_1, \ldots, y_n)$ is a system of local coordinates on the n-dimensional manifold M and f_{ij} are differentiable functions on M has exactly $n - \operatorname{rank}(f_{ij})$ functionally independent solutions. According to theorem 1, the result follows by replacing M with L and each f_{ij} with M_L^{ij}. $\square$

It is immediately obvious that since the rank of the skew-symmetric matrix M_L must be even, we have $\mathcal{N} \equiv \dim L \pmod 2$, and thus $\mathcal{N}$ and $\dim L$ have the same parity. The following result of [21] identifies those Lie algebras that have only polynomial invariants or only rational invariants.

Theorem 4.

(a) *If* L *is algebraic, then* $\tau' = \dim L - R(L)$.
(b) *If* L *is algebraic and* $D(\mathfrak{A})^I = D(\mathfrak{A}^I)$, *then* $\tau = \dim L - R(L)$.

In other words, invariants of algebraic Lie algebras can always all be chosen to be rational functions, and if in addition $D(\mathfrak{A})^I = D(\mathfrak{A}^I)$, i.e. if any rational Casimir invariant is the quotient of two Casimir invariants, then all invariants of L are polynomial functions. This sufficient condition holds for any nilpotent Lie algebra [24] and for any semisimple Lie algebra [25]. However, we cannot always find a fundamental set of invariants for nilpotent Lie algebras which is also an integrity basis in the sense that any polynomial invariant can be expressed as a polynomial function of them, as is the case for semisimple Lie algebras. Nevertheless, we know that the invariants of nilpotent Lie algebras can all be chosen to be homogeneous polynomials and consequently we shall only focus on non-nilpotent solvable Lie algebras. Their invariants generally involve logarithmic terms or functions in arctan in a manner which is not yet clearly specified [2, 3].

3. Determination of the invariants

Consider on the finite-dimensional non-nilpotent solvable Lie algebra L a vector space decomposition of the form

$$L = \mathcal{M} \oplus E \tag{3.1}$$

where $\mathcal{M}$ is the nilradical of L and E is any complement subspace of $\mathcal{M}$ in L. The generators of the subspace E are said to be linearly nil-independent, since no non-trivial linear combination of them can be ad-nilpotent. Let

$$\mathcal{A} = \{N_1, \ldots, N_r; X_1, \ldots, X_k\} \tag{3.2}$$

be a basis of L, where $\{N_1, \ldots, N_r\}$ is a basis of $\mathcal{M}$ and $\{X_1, \ldots, X_k\}$ is a basis of E. Thus $\dim \mathcal{M} = r$ and $\dim E = k$. To this basis, we associate a coordinate system of the form

$$\mathcal{S} = (n_1, \ldots, n_r; x_1, \ldots, x_k). \tag{3.3}$$

If $(v_1, \ldots, v_n)$ is any coordinate system corresponding to a basis $\{V_1, \ldots, V_n\}$ of L, by an abuse of notation we set $[v_i, v_j] = \sum_k c_{ij}^k v_k$, where the c_{ij}^k's $(i, j, k = 1, \ldots, n)$ are as usual the structure constants of L in the given basis. In this case the infinitesimal generators $\tilde{V}_i$ are given by

$$\tilde{V}_i = -\sum_j [v_i, v_j]\partial_{v_j}. \tag{3.4}$$

Since $[L, L] \subset \mathcal{M}$, it follows that in terms of the coordinate systems (3.3), $[v_i, v_j] = f_{ij}(n_1, \ldots, n_r)$ for all $i, j = 1, \ldots, n$. Thus f_{ij} does not depend on the x_j's, $(j = 1, \ldots, k)$. This remark applies to the invariants of L as well when $\mathcal{M}$ is Abelian.

Theorem 5. *With the notation of equations (3.2) and (3.3), if $\mathcal{M}$ is Abelian, then for any invariant F of L,*

$$\partial_{x_j} \cdot F = 0 \qquad (j = 1, \ldots, k) \tag{3.5}$$

so that $F = F(n_1, \ldots, n_r)$. In particular, the invariants are all determined by the reduced system of PDEs

$$\tilde{X}_j \cdot F = 0 \qquad (j = 1, \ldots, k)$$

and their determination does not depend on the $[E, E]$ type commutation relations.

This theorem significantly reduces the computation of the invariants and is proven in details in [3] for complex Lie algebras and in [4] for any Lie algebra over a field of characteristic zero. Denote by A^u the matrix of $\mathrm{ad}_{\mathcal{M}} X_u$, the restriction to $\mathcal{M}$ of the adjoint operator $\mathrm{ad}\, X_u$ $(u = 1, \ldots, k)$. In our determination of the invariants we shall also be guided by the following result of [4] related to their number.

Theorem 6.

(a) If all elements of $\{A^1, \ldots, A^k\}$ are simultaneously triangularizable, then $\mathcal{N} \leqslant r$.
(b) If the nilradical is Abelian, then $\mathcal{N} = 2r - n$.

Solutions to the system (2.1) of PDEs defining the invariants will be obtained by solving the corresponding system of characteristic equations

$$\frac{\mathrm{d}v_1}{f_{i1}} = \frac{\mathrm{d}v_2}{f_{i2}} = \cdots = \frac{\mathrm{d}v_n}{f_{in}} \qquad (i = 1, \ldots, n)$$

where $f_{ij} = f_{ij}(n_1, \ldots, n_r) = [v_i, v_j]$, and by using at each step as new variables the elements of the last fundamental set of invariants obtained. One example is treated in the next section.

Our determination of the invariants is based on the list of all equivalence classes of non-nilpotent solvable Lie algebras of dimension six having a four-dimensional nilradical, recently given by Turkowski [15]. This list completes the classification started by Mubarakzyanov [16] of all solvable Lie algebras of dimension six. The classification of nilpotent Lie algebras of dimension six was realized by Morozov [17], and their invariants are listed in [2]. In our list of solvable Lie algebras, the algebra $N_{6j}^{\alpha\beta\cdots}$ means the jth algebra of dimension six. The superscripts, if any, give the values of the continuous parameters on which the algebra depends. Restrictions on the range of the parameters are to avoid double counting and algebraic decompositions. A similar notation has been used by other authors (see, e.g., [2, 15]).

4. Solvable Lie algebras with Abelian nilradicals

In keeping with the notation of the previous section, and as stipulated by theorem 5, all the invariants are determined in the case of an Abelian nilradical by the reduced system of linear first-order partial differential equations

$$\tilde{X}_1 \cdot F = 0 \qquad \tilde{X}_2 \cdot F = 0. \tag{4.1}$$

In tables 1 and 2, we present the invariants of all non-nilpotent solvable Lie algebras of dimension six having an Abelian nilradical, together with their commutation relations. There are 27 of them, 18 of which have a centre of dimension zero. All of these Lie algebras depend on one or more continuous parameters except for the algebras $N_{6,8}$, $N_{6,18}$ and $N_{6,24}$.

To illustrate the method of determination of the invariants that we apply here, we give the solution to (4.1) for the algebra $N_{6,16}^{\alpha\beta}$. The infinitesimal generators $\tilde{X}_1$ and $\tilde{X}_2$ have the form

$$\tilde{X}_1 = -n_2\partial_{n_1} - (\alpha n_3 + n_4)\partial_{n_3} - (-n_3 + \alpha n_4)\partial_{n_4} \tag{4.2}$$

$$\tilde{X}_2 = -n_1\partial_{n_1} - n_2\partial_{n_2} - \beta n_3\partial_{n_3} - \beta n_4\partial_{n_4}. \tag{4.3}$$

We readily note in this case that $\mathrm{ad}_{\mathcal{M}}\, X_2 = \mathrm{diag}\{1, 1, \beta, \beta\}$ acts diagonally, which makes the solutions to $\tilde{X}_2 \cdot F = 0$ relatively simple. The corresponding characteristic equation

$$\frac{\mathrm{d}n_1}{n_1} = \frac{\mathrm{d}n_2}{n_2} = \frac{\mathrm{d}n_3}{\beta n_3} = \frac{\mathrm{d}n_4}{\beta n_4}$$

has the three functionally independent solutions $\xi_1 = n_1/n_2$, $\xi_2 = n_2^{\beta}/n_3$ and $\xi_3 = n_3/n_4$. We have $\tilde{X}_1(\xi_1) = 1$, $\tilde{X}_1(\xi_2) = -\xi_2[\alpha + (1/\xi_3)]$ and $\tilde{X}_1(\xi_3) = 1 + \xi_3^2$. Hence in terms of the new variables $\{\xi_1, \xi_2, \xi_3\}$, we have

$$\tilde{X}_1 = \partial_{\xi_1} - \xi_2[\alpha + (1/\xi_3)]\partial_{\xi_2} + (1 + \xi_3^2)\partial_{\xi_3}.$$

Solving the corresponding characteristic equation yields the two functionally independent solutions

$$\mathcal{I}_1 = \frac{n_1}{n_2} - \arctan\frac{n_3}{n_4}$$

$$\mathcal{I}_2 = \alpha \arctan\frac{n_3}{n_4} - \frac{1}{2}\log(n_3^2 + n_4^2) + \beta \log n_2.$$

Table 1. Invariants of solvable Lie algebras of dimension six that contain the Abelian nilradical of dimension four and the centre of dimension zero.

Name	Non-zero commutation relations		Invariants
$N_{6,1}^{\alpha\beta\gamma\delta}$	$[X_1, N_1] = \alpha N_1$	$[X_2, N_1] = \beta N_1$	$\mathcal{I}_1 = n_3^{\beta} n_4^{\alpha}/n_1$
$\alpha\beta \neq 0$	$[X_1, N_2] = \gamma N_2$	$[X_2, N_2] = \delta N_2$	$\mathcal{I}_2 = n_3^{(\beta\gamma-\delta\alpha)} n_2^{\alpha}/n_1^{\gamma}$
$\gamma^2 + \delta^2 \neq 0$	$[X_1, N_4] = N_4$	$[X_2, N_3] = N_3$	
$N_{6,2}^{\alpha\beta\gamma}$	$[X_1, N_1] = \alpha_1 N_1$	$[X_2, N_1] = \beta N_1$	$\mathcal{I}_1 = \alpha n_3/n_4 + \log(n_4^{\beta}/n_1)$
$\alpha^2 + \beta^2 \neq 0$	$[X_1, N_2] = N_2$	$[X_2, N_2] = \gamma N_2$	$\mathcal{I}_2 = n_3/n_4 + \log(n_4^{\gamma}/n_2)$
	$[X_1, N_3] = N_4$	$[X_2, N_3] = N_3$	
		$[X_2, N_4] = N_4$	
$N_{6,3}^{\alpha}$	$[X_1, N_1] = N_1$	$[X_2, N_1] = \alpha N_1 + N_2$	$\mathcal{I}_1 = n_3/n_4 - \log n_2$
	$[X_1, N_2] = N_2$	$[X_2, N_2] = \alpha N_2$	$\mathcal{I}_2 = n_1/n_2 - \log n_4$
	$[X_1, N_3] = N_4$	$[X_2, N_3] = N_3$	
	$[X_2, N_4] = N_4$		
$N_{6,4}^{\alpha\beta}$	$[X_1, N_1] = N_1$	$[X_2, N_1] = N_2$	$\mathcal{I}_1 = \alpha \arctan(n_1/n_2) - \log n_4$
$\alpha \neq 0$	$[X_1, N_2] = N_2$	$[X_2, N_2] = -N_1$	$\mathcal{I}_2 = \beta \arctan(n_1/n_2) + \frac{1}{2}\log(n_1^2 + n_2^2) - n_3/n_4$
	$[X_1, N_3] = N_4$	$[X_2, N_3] = \alpha N_3 + \beta N_4$	
		$[X_2, N_4] = \alpha N_4$	
$N_{6,5}^{\alpha\beta}$	$[X_1, N_1] = \alpha N_1$	$[X_2, N_1] = \beta N_1$	$\mathcal{I}_1 = n_2^{\beta} n_4^{\alpha}/n_1$
$\alpha \neq 0$	$[X_1, N_3] = N_3 + N_4$	$[X_2, N_2] = N_2$	$\mathcal{I}_2 = n_3/n_4 - \log n_4$
	$[X_1, N_4] = N_4$		
$N_{6,6}^{\alpha\beta}$	$[X_1, N_1] = \alpha N_1$	$[X_2, N_1] = N_1 + N_2$	$\mathcal{I}_1 = \log n_4 - n_3/n_4 + \beta \log n_2/n_4^{\alpha}$
$\alpha^2 + \beta^2 \neq 0$	$[X_1, N_2] = \alpha N_2$	$[X_2, N_2] = N_2$	
	$[X_1, N_3] = N_3 + N_4$	$[X_2, N_3] = \beta N_4$	$\mathcal{I}_2 = n_1/n_2 - \log n_2/n_4^{\alpha}$
	$[X_1, N_4] = N_4$		
$N_{6,7}^{\alpha\beta\gamma}$	$[X_1, N_1] = \alpha N_1$	$[X_2, N_1] = \gamma N_1 + N_2$	$\mathcal{I}_1 = \log n_4 + \beta \arctan(n_1/n_2) - n_3/n_4$
$\alpha^2 + \beta^2 \neq 0$	$[X_1, N_2] = \alpha N_2$	$[X_2, N_2] = -N_1 + \gamma N_2$	
	$[X_1, N_3] = N_3 + N_4$	$[X_2, N_3] = \beta N_4$	$\mathcal{I}_2 = \alpha \log n_4 + \gamma \arctan(n_1/n_2) - \frac{1}{2}\log(n_1^2 + n_2^2)$
	$[X_1, N_4] = N_4$		
$N_{6,8}$	$[X_1, N_1] = N_1$	$[X_2, N_2] = N_2$	$\mathcal{I}_1 = \dfrac{n_3}{n_4} - \log n_4$
	$[X_1, N_2] = N_4$	$[X_2, N_3] = N_3 + N_4$	$\mathcal{I}_2 = \dfrac{n_2}{n_4} - \log n_1$
		$[X_2, N_4] = N_4$	
$N_{6,9}^{\alpha}$	$[X_1, N_1] = N_1$	$[X_2, N_2] = N_2 + N_3$	$\mathcal{I}_1 = \dfrac{n_3}{n_4} - \alpha \log n_4$
	$[X_1, N_2] = N_4$	$[X_2, N_3] = N_3 + \alpha N_4$	$\mathcal{I}_2 = \dfrac{2\alpha n_2 n_4 - n_3^2}{2n_4^2} - \alpha \log n_1$
		$[X_2, N_4] = N_4$	

It then follows from theorem 5 and part (a) of theorem 1 that $\{\mathcal{I}_1, \mathcal{I}_2\}$ is a fundamental set generating the invariants of the Lie algebra $N_{6,16}^{\alpha\beta}$.

Table 1. Continued.

Name	Non-zero Commutation relations		Invariants
$N_{6,10}^{\alpha\beta}$	$[X_1, N_1] = \alpha N_1$	$[X_2, N_1] = N_1$	$\mathcal{I}_1 = \dfrac{n_3^2}{n_4^2} - 2\dfrac{n_2}{n_4} + \log n_4^{2\beta}$
	$[X_1, N_2] = N_2 + \beta N_4$	$[X_2, N_2] = N_3$	$\mathcal{I}_2 = \dfrac{n_3}{n_4} + \log \dfrac{n_4^{\alpha}}{n_1}$
	$[X_1, N_3] = N_3$	$[X_2, N_3] = N_4$	
	$[X_1, N_4] =$		
$N_{6,11}^{\alpha}$	$[X_1, N_1] = N_2$	$[X_2, N_1] = N_1$	$\mathcal{I}_1 = \dfrac{n_1}{n_2} - \log \dfrac{n_4}{n_2^{\alpha}}$
	$[X_1, N_3] = N_3 + N_4$	$[X_2, N_1] = N_2$	$\mathcal{I}_2 = \dfrac{n_3}{n_4} - \log \dfrac{n_4}{n_2^{\alpha}}$
	$[X_1, N_4] = N_4$	$[X_2, N_3] = \alpha N_3$	
		$[X_2, N_4] = \alpha N_4$	
$N_{6,12}^{\alpha\beta}$	$[X_1, N_1] = N_1 + N_2$	$[X_2, N_1] = \alpha N_2 + N_3 - \beta N_4$	$\mathcal{I}_1 = \dfrac{n_2 n_3 - n_1 n_4}{n_2^2 + n_4^2} + \arctan \dfrac{\beta n_2 + \alpha n_4}{\alpha n_2 + \beta n_4}$
	$[X_1, N_2] = N_2$	$[X_2, N_2] = N_4$	$\mathcal{I}_2 = \beta \log(n_2^2 + n_4^2)$
	$[X_1, N_3] = N_3 + N_4$	$[X_2, N_3] = -N_1 + \beta N_2 + \alpha N_4$	$+2\dfrac{\alpha(n_2 n_3 - n_1 n_4) - \beta(n_1 n_2 + n_3 n_4)}{n_2^2 + n_4^2}$
	$[X_1, N_4] = N_4$	$[X_2, N_4] = -N_2$	if $(\alpha^2 + \beta^2) \neq 0$
			$= \log(n_2^2 + n_4^2) + \dfrac{2(n_1 n_2 + n_3 n_4)}{n_2^2 + n_4^2}$
			if $\alpha^2 + \beta^2 = 0$
$N_{6,13}^{\alpha\beta\gamma\delta}$	$[X_1, N_1] = \alpha N_1$	$[X_2, N_2] = \beta N_1$	$\mathcal{I}_1 = \log n_1 + (\beta - 1)\log n_3$
$\alpha^2 + \beta^2 \neq 0$	$[X_1, N_2] = \gamma N_2$	$[X_2, N_2] = \delta N_2$	$+\alpha \arctan \dfrac{n_3}{n_4} - \dfrac{\beta}{2}\log(n_3^2 + n_4^2)$
	$[X_1, N_3] = N_4$	$[X_2, N_3] = N_3$	$\mathcal{I}_2 = \log n_2 + (\delta - 1)\log n_3$
		$[X_2, N_4] = N_4$	$+\gamma \arctan \dfrac{n_4}{n_3} - \dfrac{\delta}{2}\log(n_3^2 + n_4^2)$
$N_{6,14}^{\alpha\beta\gamma}$	$[X_1, N_1] = N_1$	$[X_2, N_1] = \beta N_1$	$\mathcal{I}_1 = \alpha \arctan \dfrac{n_4}{n_3} + \log(n_1/n_2^{\beta})$
$\alpha\beta \neq 0$	$[X_1, N_3] = \gamma N_3 + N_4$	$[X_2, N_1] = N_2$	$\mathcal{I}_2 = \gamma \arctan \dfrac{n_4}{n_3} + \dfrac{1}{2}\log(n_3^2 + n_4^2)$
	$[X_1, N_4] = -N_3 + \gamma N_4$		
$N_{6,15}^{\alpha\beta\gamma\delta}$	$[X_1, N_1] = N_1$	$[X_2, N_1] = \gamma N_1 + N_2$	$\mathcal{I}_1 = 2(\alpha\gamma - \delta)\arctan \dfrac{n_1}{n_2} + \log \dfrac{n_3^2 + n_4^2}{(n_1^2 + n_2^2)^{\alpha}}$
$\beta \neq 0$	$[X_1, N_2] = N_2$	$[X_2, N_2] = -N_1 + \gamma N_2$	
	$[X_1, N_3] = \alpha N_3 + \beta N_4$	$[X_2, N_3] = \delta N_3$	$\mathcal{I}_2 = 2\beta\gamma \arctan \dfrac{n_1}{n_2} - \beta \log(n_1^2 + n_2^2)$
	$[X_1, N_4] = -\beta N_3 + \alpha N_4$	$[X_2, N_4] = \delta N_4$	
$N_{6,16}^{\alpha\beta}$	$[X_1, N_1] = N_2$	$[X_2, N_1] = N_1$	$\mathcal{I}_1 = \dfrac{n_1}{n_2} - \arctan \dfrac{n_3}{n_4}$
	$[X_1, N_3] = \alpha N_3 + N_4$	$[X_2, N_2] = N_2$	$\mathcal{I}_2 = \alpha \arctan \dfrac{n_3}{n_4} - \dfrac{1}{2}\log(n_3^2 + n_4^2) + \beta \log n_2$
	$[X_1, N_4] = -N_3 + \alpha N_4$	$[X_2, N_3] = \beta N_3$	
		$[X_2, N_4] = \beta N_4$	

Table 1. Continued.

Name	Non-zero commutation relations		Invariants
$N_{6,17}^{\alpha}$	$[X_1, N_1] = \alpha N_1 + N_2$	$[X_2, N_3] = N_3$	$\mathcal{I}_1 = \dfrac{n_1}{n_2} - \arctan\dfrac{n_3}{n_4}$
	$[X_1, N_2] = \alpha N_2$	$[X_2, N_4] = N_4$	$\mathcal{I}_2 = \alpha\dfrac{n_1}{n_2} - \log n_2$
	$[X_1, N_3] = N_4$		
	$[X_1, N_4] = -N_3$		
$N_{6,18}^{\alpha\beta\gamma}$	$[X_1, N_1] = N_2$	$[X_2, N_1] = N_1$	$\mathcal{I}_1 = \arctan\dfrac{n_4}{n_3} + \beta\arctan\dfrac{n_1}{n_2}$
$\beta \neq 0$	$[X_1, N_2] = -N_1$	$[X_2, N_2] = N_2$	$\mathcal{I}_2 = \dfrac{2\alpha}{\beta}\arctan\dfrac{n_3}{n_4} + \log\dfrac{(n_1^2+n_2^2)^{\gamma}}{n_3^2+n_4^2}$
	$[X_1, N_3] = \alpha N_3 + \beta N_4$	$[X_2, N_3] = \gamma N_3$	
	$[X_1, N_4] = -\beta N_3 + \alpha N_4$	$[X_2, N_4] = \gamma N_4$	
$N_{6,19}$	$[X_1, N_1] = N_2 + N_3$	$[X_2, N_1] = N_1$	$\mathcal{I}_1 = \dfrac{n_1n_3 + n_2n_4}{n_3^2+n_4^2} + \arctan\dfrac{n_4}{n_3}$
	$[X_1, N_2] = -N_1 + N_4$	$[X_2, N_2] = N_2$	$\mathcal{I}_2 = \dfrac{n_1n_4 - n_2n_3}{n_3^2+n_4^2}$
	$[X_1, N_3] = N_4$	$[X_2, N_3] = N_3$	
	$[X_1, N_4] = -N_3$	$[X_2, N_4] = N_4$	

It appears from the tables that all solvable Lie algebras with Abelian nilradicals have a fundamental set of invariants consisting of two functions. This is in accordance with part (b) of theorem 6 which asserts that the number of invariants in such cases is $\mathcal{N} = 2r - n = 2$. Now, denote by $\mathcal{A}_2$ the non-nilpotent solvable Lie algebra of dimension two with commutation relations $[X, N] = N$ and by $\mathfrak{g}$ the solvable algebra of dimension four with structure $[X_1, N_1] = N_1$, $[X_1, N_2] = N_1$, $[X_2, N_1] = -N_2$, $[X_2, N_2] = N_2$, where N_1 and N_2 are the generators of the corresponding nilradical. Then we have the following characterization of solvable Lie algebras with Abelian nilradicals that admit no non-trivial invariant function.

Theorem 7. *Let L be the non-nilpotent solvable Lie algebra of dimension n over a field $\mathbb{K}$ of characteristic zero and having an Abelian nilradical $\mathcal{M}$ of dimension r.*

(a) When $\mathbb{K} = \mathbb{C}$, then L has no non-trivial invariant if and only if

$$L = r\mathcal{A}_2. \tag{4.4}$$

(b) When $\mathbb{K} = \mathbb{R}$, then L has no non-trivial invariant if and only if

$$L = s\mathfrak{g} \oplus (n - r - 2s)\mathcal{A}_2 \tag{4.5}$$

where s is the number of pairs of distinct complex conjugate roots of L.

Proof. Clearly, all the Lie algebras of (4.4) and (4.5) have Abelian nilradicals and satisfy the condition $2\dim\mathcal{M} = \dim L$, and hence by part (b) of theorem 6 they have no non-trivial invariant. Thus we only need to prove that if a solvable Lie algebra with Abelian nilradical has no non-trivial invariant, then it is of the stated form. Since the nilradical of L is Abelian, if L has no non-trivial invariant, then by part (b) of theorem 6 we must have $2r = n$. Moreover, the centre $Z(L)$ of L has dimension zero and hence the condition

Table 2. Invariants of solvable Lie algebras of dimension six that contain Abelian nilradical of dimension four and the centre of dimension one.

Name	Non-zero commutation relations		Invariants
$N_{6,20}^{\alpha\beta}$	$[X_1, N_2] = \alpha N_2$	$[X_2, N_2] = \beta N_2$	$\mathcal{I}_1 = n_1$
$\alpha^2 + \beta^2 \neq 0$	$[X_1, N_4] = N_4$	$[X_2, N_3] = N_3$	$\mathcal{I}_2 = \dfrac{n_3^{\beta} n_4^{\alpha}}{n_2}$
	$[X_1, X_2] = N_1$		
$N_{6,21}^{\alpha}$	$[X_1, N_2] = N_2$	$[X_2, N_2] = \alpha N_2$	$\mathcal{I}_1 = n_1$
	$[X_1, N_2] = N_2$	$[X_2, N_2] = \alpha N_2$	$\mathcal{I}_2 = \dfrac{n_3}{n_4} - \log n_2$
	$[X_1, X_2] = N_1$	$[X_2, N_4] = N_4$	
$N_{6,22}^{\alpha\epsilon}$	$[X_1, N_1] = N_1$	$[X_2, N_1] = \alpha N_1$	$\mathcal{I}_1 = n_4$
$\epsilon = 0, 1$	$[X_1, N_3] = N_4$	$[X_2, N_2] = N_2$	$\mathcal{I}_2 = \dfrac{n_3}{n_4} + \log \dfrac{n_2^{\alpha}}{n_1}$
$\alpha^2 + \epsilon^2 \neq 0$	$[X_1, X_2] = \epsilon N_3$		
$N_{6,23}^{\alpha\epsilon}$	$[X_1, N_1] = N_1$	$[X_2, N_1] = N_2$	$\mathcal{I}_1 = n_4$
$\epsilon = 0, 1$	$[X_1, N_2] = N_2$	$[X_2, N_2] = -N_1$	$\mathcal{I}_2 = \dfrac{n_3}{n_4} - \dfrac{1}{2}\log(n_1^2 + n_2^2) - \alpha \arctan \dfrac{n_1}{n_2}$
	$[X_1, N_3] = N_4$	$[X_2, N_3] = \alpha N_4$	
	$[X_1, X_2] = \epsilon N_3$		
$N_{6,24}$	$[X_1, N_3] = N_3 + N_4$	$[X_2, N_2] = N_2$	$\mathcal{I}_1 = n_1$
	$[X_1, N_4] = N_4$		$\mathcal{I}_2 = \dfrac{n_3}{n_4} - \log n_4$
	$[X_1, X_2] = N_1$		
$N_{6,25}^{\alpha\beta}$ $(*)$	$[X_1, N_2] = \alpha N_2$	$[X_2, N_1] = \beta N_2$	$\mathcal{I}_1 = \alpha \arctan \dfrac{n_3}{n_4} - \log n_2$
$\alpha^2 + \beta^2 \neq 0$	$[X_1, N_3] = N_4$	$[X_2, N_3] = N_3$	$\mathcal{I}_2 = \dfrac{n_1}{n_2} - \dfrac{\beta}{2}\log(n_4^2 + n_3^2)$
	$[X_1, N_4] = -N_3$	$[X_2, N_4] = N_4$	
	$[X_1, X_2] = N_1$		
$N_{6,26}^{\alpha}$	$[X_1, N_3] = \alpha N_3 + N_4$	$[X_2, N_2] = N_2$	$\mathcal{I}_1 = n_1$
	$[X_1, N_4] = -N_3 + \alpha N_4$		$\mathcal{I}_2 = \alpha \arctan \dfrac{n_4}{n_3} + \dfrac{1}{2}\log(n_3^2 + n_4^2)$
	$[X_1, X_2] = N_1$		
$N_{6,27}^{\epsilon}$	$[X_1, N_1] = N_2$	$[X_2, N_3] = N_3$	$\mathcal{I}_1 = n_2$
$\epsilon = 0, 1$	$[X_1, N_3] = N_4$	$[X_2, N_4] = N_4$	$\mathcal{I}_2 = \dfrac{n_1}{n_2} - \arctan \dfrac{n_3}{n_4}$
	$[X_1, N_4] = -N_3$		
	$[X_1, X_2] = \epsilon N_1$		

$\dim \mathcal{M} = \frac{1}{2}[\dim L + \dim Z(L)]$ holds. Consequently, when $\mathbb{K} = \mathbb{C}$, theorem 2 of [16] asserts that L has the structure $[X_i, N_i] = N_i,\ [N_i, N_j] = 0,\ [X_i, X_j] \in Z(L)\ (i, j = 1, \ldots, n - r)$. That is, $L = r\mathcal{A}_2$, and this completes the proof of part (a). For part (b), when $\mathbb{K} = \mathbb{R}$, theorem 3 of [16] asserts that L contains s subalgebras with the structure of $\mathfrak{g}$ and $(n-r)-2s$ non-nilpotent subalgebras with the structure of $\mathcal{A}_2$, and the commutator $[X_i, X_j]$ of all linearly nil-independent elements are in the centre $Z(L)$. Since $Z(L) = \{0\}$, this means that L is of the form given by (4.5), and this completes the proof of the theorem. □

The functions defining these invariants are far from being Casimir operators, since they generally involve log- and arctan-type functions. The only cases in which they are rational occur when both $\mathrm{ad}_{\mathcal{M}} X_1$ and $\mathrm{ad}_{\mathcal{M}} X_2$ act diagonally, and this only happens with the algebras $N_{6,1}^{\alpha\beta\gamma\delta}$ and $N_{6,20}^{\alpha\beta}$. However, these rational invariants can become polynomial for certain values of the parameters on which these two Lie algebras depend. For instance, $N_{6,1}^{\alpha\beta\gamma\delta}$ has a fundamental set consisting of Casimir operators if the sequence of numbers $\{\alpha, \beta, -1\}$ on one hand and $\{\beta\gamma-\delta\alpha, \alpha, -\gamma\}$, on the other hand, are of the same signs. Thus the only possibilities where the invariants can be chosen to be rational functions and eventually Casimir invariants correspond to the case where the operators $\mathrm{ad}_{\mathcal{M}} X_u$ $(u = 1, 2)$ are all diagonal. Conversely, when these operators are all diagonal, the invariants can always be chosen to be rational functions by a result of [3]. However, the question is still open as to whether in the general case of an Abelian nilradical (with $n = \dim L$ arbitrary), a fundamental set consisting of rational functions can occur only when the operators $\mathrm{ad}_{\mathcal{M}} X_u$ are all simultaneously diagonal, and if the corresponding Lie algebra is algebraic as stipulated by the sufficient condition of theorem 4.

We also note from the tables that when the dimension of the centre is one, one of the two invariants is always polynomial. This is obvious since the non-zero generator of the centre can be identified with a Casimir operator. An exception apparently occurs with the Lie algebra $N_{6,25}^{\alpha\beta}$. However, this in reality is not an exception as such, since we can easily verify that this Lie algebra has rather a zero-dimensional centre. It is therefore by a simple mistake of the author of [15] that this Lie algebra (indicated in the table with a (*)) is listed among the solvable Lie algebras with a centre of dimension one.

5. Solvable Lie algebra with non-Abelian nilradicals

When the nilradical is not Abelian, it is necessary for the determination of the invariants to solve for each Lie algebra the system of six equations $\tilde{N}_i \cdot F = 0$ $(i = 1, \ldots, 4)$ and $\tilde{X}_j \cdot F = 0$ $(j = 1, 2)$. However, most of these equations generally degenerate into trivial equations, or simple conditions, and the reduced system obtained is solved with the method presented in the previous section.

There are 13 non-nilpotent solvable Lie algebras with non-Abelian nilradicals in the classification of [15]. Of these Lie algebras, eight have no invariants while the remaining five of them have two invariants each. We remark that the number 2 appears to be an upper bound for the number of invariants. However, it is easy to see that in all these cases, the matrices of the operators $\mathrm{ad}_{\mathcal{M}} X_1$, $\mathrm{ad}_{\mathcal{M}} X_2$, can be simultaneous put into a triangular form, so that the upper bound obtained can been seen as an immediate application of part (a) of theorem 6 which asserts that the maximal number of functionally independent invariants cannot exceed four, the dimension of the nilradical. It would be useful to characterize all of those solvable Lie algebras with non-Abelian nilradicals that do not have non-trivial invariants functions, as we did in theorem 7 for the case of Abelian nilradicals. It is worth remarking that in contrast to the case of non-nilpotent solvable Lie algebras, the number of invariants of semisimple and nilpotent Lie algebras is always non-zero. Indeed, for semisimple Lie algebras this number is the dimension of the Cartan subalgebra, which in this case corresponds to the maximal toral subalgebra. Therefore, this number is always non-zero on any field of characteristic zero [9].

Table 3. Invariants of real solvable Lie algebra that contains the non-Abelian nilradical $A_{3,1} \oplus A_1$ and the centre of dimension zero.

Name	Non-zero Commutation relations		Invariants
$N_{6,29}^{\alpha,\beta}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_2$	None
$\alpha^2 + \beta^2 \neq= 0$	$[X_1, N_1] = N_1$	$[X_2, N_1] = N_1$	
	$[X_2, N_1] = N_1$	$[X_1, N_4] = \alpha N_4$	
		$[X_2, N_4] = \beta N_4$	
$N_{6,30}^{\alpha}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_2$	None
	$[X_1, N_1] = 2N_1$	$[X_2, N_2] = N_3$	
	$[X_1, N_4] = \alpha N_4$	$[X_1, N_3] = N_3$	
		$[X_2, N_4] = N_4$	
$N_{6,31}$	$[N_2, N_3] = N_1$	$[X_1, N_3] = -N_3$	$\mathcal{I}_1 = x_1 + \dfrac{n_2 n_3}{n_1}$
	$[X_1, N_2] = N_2$	$[X_2, N_4] = N_1 + N_4$	$\mathcal{I}_2 = \dfrac{n_4}{n_1} - \log n_1$
	$[X_2, N_3] = N_3$	$[X_2, N_1] = N_1$	
$N_{6,32}^{\alpha}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_2$	None
	$[X_1, N_4] = N_1$	$[X_2, N_1] = N_1$	
	$[X_2, N_3] = (1 - \alpha)N_3$	$[X_2, N_4] = N_4$	
	$[X_1, N_3] = -N_3$	$[X_2, N_2] = \alpha N_2$	
$N_{6,33}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_2$	None
	$[X_1, N_1] = N_1$	$[X_2, N_4] = N_4$	
	$[X_2, N_3] = N_3 + N_4$	$[X_2, N_1] = N_1$	
$N_{6,34}^{\alpha}$	$[N_2, N_3] = N_1$	$[X_1, N_1] = N_1$	None
	$[X_1, N_3] = N_4$	$[X_2, N_1] = (1 + \alpha)N_1$	
	$[X_2, N_2] = \alpha N_2$	$[X_2, N_3] = N_3$	
	$[X_1, N_2] = N_2$	$[X_2, N_4] = N_4$	
$N_{6,35}^{\alpha\beta}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_3$	None
$\alpha + \beta \neq 0$	$[X_1, N_4] = \alpha n_4$	$[X_2, N_1] = 2N_1$	
	$[X_2, N_3] = N_3$	$[X_2, N_4] = \beta N_4$	
	$[X_1, N_3] = -N_2$	$[X_2, N_2] = N_2$	
$N_{6,36}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_3$	$\mathcal{I}_1 = \dfrac{n_2^2 + n_3^2}{n_1} + 2x_1$
	$[X_2, N_1] = 2N_1$	$[X_2, N_2] = N_2$	$\mathcal{I}_2 = \dfrac{2n_4}{n_1} - \log n_1$
	$[X_2, N_4] = N_1 + 2N_4$	$[X_1, N_3] = -N_2$	
		$[X_2, N_3] = N_3$	
$N_{6,37}^{\alpha}$	$[N_2, N_3] = N_1$	$[X_1, N_3] = -N_2$	None
	$[X_1, N_2] = N_3$	$[X_2, N_2] = N_2 + \alpha N_3$	
	$[X_2, N_1] = 2N_1$	$[X_2, N_4] = 2N_4$	
	$[X_2, N_3] = -\alpha N_2 + N_3$	$[X_1, N_4] = N_1$	

Table 4. Invariants of real solvable Lie algebra that contains the non-Abelian nilradical $A_{3,1} \oplus A_1$ and the centre of dimension one.

Name	Non-zero commutation relations		Invariants
$N_{6,38}$	$[N_2, N_3] = N_1$	$[X_2, N_1] = N_1$	$\mathcal{I}_1 = n_4$
	$[X_1, N_1] = N_1$	$[X_2, N_3] = N_3$	$\mathcal{I}_2 = \dfrac{n_2n_3 - n_1(x_1 - x_2)}{n_1n_4} + \log n_1$
	$[X_1, N_2] = N_2$	$[X_1, X_2] = N_4$	
$N_{6,39}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_3$	$\mathcal{I}_1 = n_4$
	$[X_2, N_1] = 2N_1$	$[X_2, N_2] = N_2$	$\mathcal{I}_2 = \dfrac{n_2^2 + n_3^2 + 2n_1x_1}{n_1n_4} + \log n_1$
	$[X_1, X_2] = N_4$	$[X_1, N_3] = -N_2$	
		$[X_2, N_3] = N_3$	
$N_{6,40}$	$[N_2, N_3] = N_1$	$[X_1, N_2] = N_3$	$\mathcal{I}_1 = n_1$
	$[N_2, N_4] = N_4$	$[X_1, X_2] = N_1$	$\mathcal{I}_2 = \dfrac{1}{2n_1^2}(n_2^2 + n_3^2 + 2n_1x_1) + \log n_4$
		$[X_1, N_3] = -N_2$	

Table 5. Invariants of real solvable Lie algebra that contains the non-Abelian nilradical $A_{4,1}$.

Name	Non-zero commutation relations		Invariants
$N_{6,28}$	$[N_2, N_4] = N_1$	$[N_3, N_4] = N_2$	None
	$[X_1, N_1] = N_1$	$[X_1, N_3] = -N_3$	
	$[X_2, N_2] = N_2$	$[X_2, N_3] = 2N_3$	
	$[X_1, N_4] = N_4$	$[X_2, N_4] = -N_4$	

For nilpotent Lie algebras, this result is the consequence of the fact that the algebra always possesses a non-zero centre.

There is no algebra in tables 3–5 that have a fundamental set of invariants consisting of only polynomial or even rational invariants. However, this cannot be perceived as some characteristic of solvable Lie algebras with non-Abelian nilradicals. Consider, for example, the four-dimensional algebra $A_{4,8}$ of [2] with commutation relations $[N_2, N_3] = N_1, [N_2, X_1] = N_2, [N_3, X_1] = -N_3$. This is clearly a non-nilpotent solvable Lie algebra having the non-Abelian nilradical $\mathcal{M} = \langle N_1, N_2, N_3\rangle$. It has a fundamental set of invariants consisting of the two polynomials n_1 and $n_2n_3 - n_1x_1$. The same counter-example holds with the non-nilpotent solvable Lie algebras $A_{4,10}$, $A_{5,22}$ and $A_{5,29}$ of the same reference that all have non-Abelian nilradicals and yet a fundamental set consisting of polynomial invariants. This shows that the property for a solvable Lie algebra to possess a fundamental set of invariants consisting of Casimir operators is independent of the Abelian attribute of its nilradical.

6. Concluding remarks

By determining the invariants of all non-nilpotent solvable Lie algebras of dimension six having a nilradical of dimension four, we have answered in the actual context of all information available on solvable Lie algebras, the natural question of what the invariants of solvable Lie algebras of higher dimensions look like and how they agree with the known theorems on the invariant functions. We have not given the invariants for the 99 classes of solvable Lie algebras of dimension six with nilradicals of dimension five listed in [16]. However, we feel that this case for which dim $L/\mathcal{M} = 1$ is simpler and the corresponding invariants are quite similar to that of solvable Lie algebras of dimension five with nilradicals of dimension four computed in [2]. In particular, when the nilradical is Abelian, all invariants are determined by the sole operator $\mathrm{ad}_{\mathcal{M}}\, X_1$ and this determination is straightforward since the latter operator can be put, for example, into its Jordan canonical form. The resulting form of the invariants is given in [3].

As for the type of functions in terms of which they can be expressed, all the invariants that we have determined can be put into two categories. Indeed, in terms of the usual coordinate functions $(v_1, \dots, v_n)$, all the invariants in the case of non-Abelian nilradical have the form

$$F = \frac{P}{Q} + \log v_{i_1}^{\alpha_{i_1}} \dots v_{i_p}^{\alpha_{i_p}} + \arctan \frac{\mathcal{L}_1(v_{s_1}, v_{s_2})}{\mathcal{L}_2(v_{t_1}, v_{t_2})} + \log(v_{k_1^2} + v_{k_2^2})^a (v_{g_1}^2 + v_{g_2}^2)^b \tag{6.1}$$

where P is a homogeneous polynomial and Q is a monomial of the same degree as P or of zero degree, and $\alpha_{i_1}, \dots, \alpha_{i_p}$ $(p \leqslant n)$ and a, b are elements of $\{0, 1, \alpha_1, \dots, \alpha_t, -\alpha_1, \dots, -\alpha_t\}$, where $\alpha_1, \dots, \alpha_t$ denote the t parameters on which the Lie algebra depend; $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear functions and $(i_1, \dots, i_p; s_1, s_2; t_1, t_2, k_1, k_2, g_1, g_2 \in \{1, \dots, n\})$. For solvable Lie algebras with non-Abelian nilradicals, the functions are simpler and equation (6.1) expressing the general form of the invariants is reduced to

$$F = \frac{P}{Q} + \epsilon \log v_s \qquad (\epsilon = 0 \text{ or } 1;\ s \in \{1, \dots, n\}). \tag{6.2}$$

It is not surprising that the invariants of real solvable Lie algebras involve more complicated functions than those corresponding to complex solvable Lie algebras computed, for example, in [3] and whose general form is similar to that given by (6.2). Indeed, all real matrices of the operators $\mathrm{ad}_{\mathcal{M}}\, X_u$ that give rise to invariant functions of arctan type or to functions in

$$\log(v_{k_1}^2 + v_{k_2}^2)^a (v_{g_1}^2 + v_{g_2}^2)^b$$

triangularize over $\mathbb{C}$ to give simpler functions of the form $P/Q + \log v_{i_1}^{\alpha_1} \dots v_{i_p}^{\alpha_p}$.

The number of invariants for each Lie algebra in our tables is either zero or two. This agrees perfectly with theorem 6. However, we still need to find a characterization of all non-nilpotent solvable Lie algebras with non-Abelian nilradicals that have no non-trivial invariant. We have given this characterization in theorem 7 only for solvable Lie algebras with Abelian nilradicals.

Although it is true for all non-nilpotent Lie algebras with Abelian nilradicals for which the invariants have been computed—and including all those studied in [2, 3]—that Casimir operators may occur only if the operators $\mathrm{ad}_{\mathcal{M}}\, X_u$ act diagonally, we have shown by many examples that the existence of a fundamental set of invariants consisting of Casimir operators is not related to any Abelian property of the nilradical. It would be desirable to establish a link between the number of functionally independent Casimir invariants of solvable Lie algebras with the dimension of their Cartan subalgebras, an analogue of the result of Racah [1] for semisimple Lie algebras.

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